New Keynesian Model: Approximation

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Preliminary and incomplete
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NK Model is often analyzed in LQ—Linear-Quadratic—
framework.

► Easier to interpret.

► Easier to solve.

Technical takeaways from this lecture:

▶ Learn how to take the first-order approximation to the equilibrium conditions.

▶ Learn how to take the second-order approximation to the household's welfare.

Substantive takeaways from this lecture (I):

▶ Output today depends on the expected sum of future real rates.

▶ Inflation today depends on the expected discounted sum of future marginal costs.

Substantive takeaways from this lecture (II):

▶ Welfare today depends on the expected discounted sum of future per-period utility.

    ▶ Per-period utility depends on inflation and output volatility.

    ▶ Per-period utility also depends on the level of output if steady state is not efficient.

Plan

- First-Order Approximation of the Equilibrium Conditions

- Second-Order Approximation of the Household's Welfare

Plan

- **First-Order Approximation of the Equilibrium Conditions**

- Second-Order Approximation of the Household's Welfare

Math Background

Log-linearization $=$

First-order Taylor expansion

$+$

Log-approximation of percent changes

First-order Taylor expansion:

Let f(x) be a continuous and differentiable function of x. Then,

$$f(x) \approx f(x_0) + \frac{\partial f}{\partial x}(x_0)(x - x_0) + (\textit{higher order terms})$$

Let f_x denote $\frac{\partial f}{\partial x}$.
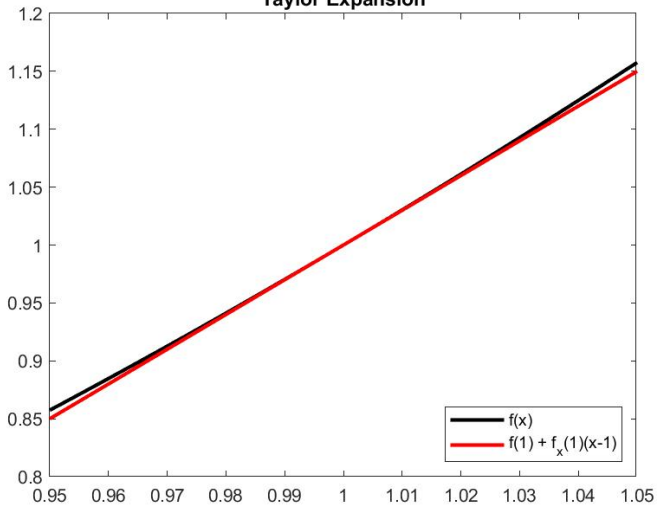
Example:

$$f(x) = x^3$$
$$f_x = 3x^2$$

Let's take the first-order Taylor expansion around $x_0 = 1$.

$$f(x) \approx f(1) + f_x(1)(x - 1) + (h.o.t.)$$
$$= 1^3 + 3 * 1^2(x - 1) + (h.o.t.)$$
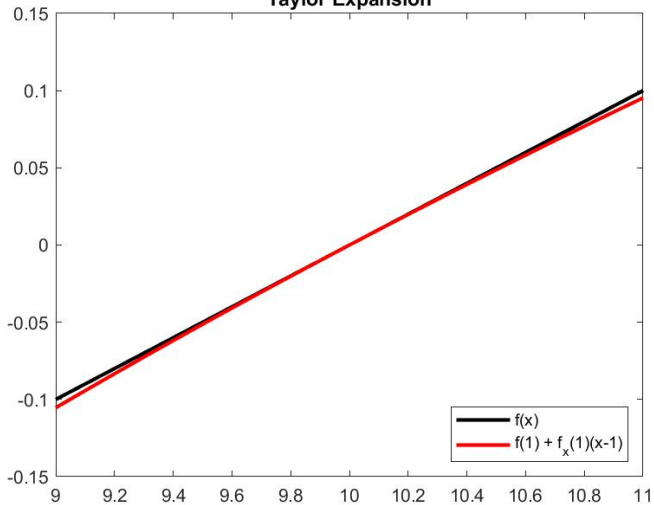$$= 1 + 3(x - 1) + (h.o.t.)$$

Taylor Expansion

Log-approximation:

$$\frac{(x - x_0)}{x_0} \approx \log(x) - \log(x_0)$$

Example:

$$\frac{(x - 10)}{x_0} \approx \log(x) - \log(10)$$

Taylor Expansion

Legend:
- f(x)
- $f(1) + f_x(1)(x-1)$

First-order Taylor expansion $+$ Log-approximation of percent changes:

$$
\begin{aligned}
f(x) &\approx f(x_0) + f_x(x_0)(x - x_0) + (h.o.t.) \\
&= f(x_0) + f_x(x_0)x_0 \frac{(x - x_0)}{x_0} + (h.o.t.) \\
&\approx f(x_0) + f_x(x_0)x_0(\log(x) - \log(x_0)) + (h.o.t.) \\
&\approx f(x_0) + f_x(x_0)x_0\hat{x} + (h.o.t.)
\end{aligned}
$$

where $\hat{x} := \log(x) - \log(x_0)$

To summarize

Log-linearization:

$$f(x) \approx f(x_0) + f_x(x_0)x_0\hat{x} + (h.o.t.)$$

where $\hat{x} := \log(x) - \log(x_0)$

Second-order Taylor expansion:

Let f(x) be a continuous and differentiable function of x. Then,

$$f(x) \approx f(x_0) + \frac{\partial f}{\partial x}(x_0)(x - x_0) + \frac{\partial^2 f}{\partial x^2}(x_0)(x - x_0)^2 + (h.o.t.)$$

Let f_x and f_{xx} denote $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$, respectively.
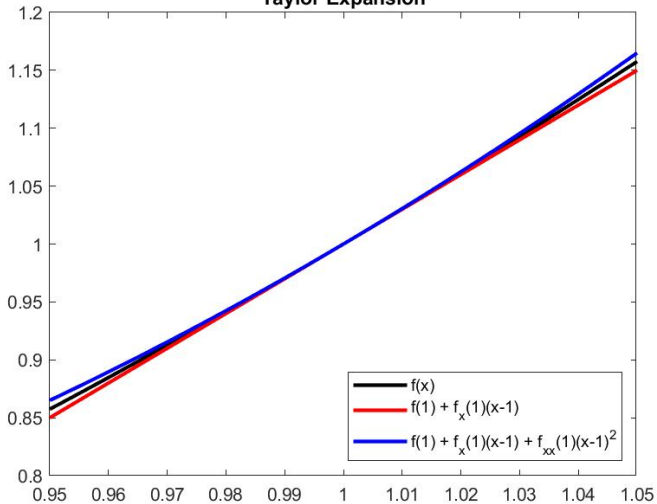
Example:

$$f(x) = x^3$$
$$f_x = 3x^2$$
$$f_{xx} = 6x$$

Let's take the second-order Taylor expansion around $x_0 = 1$.

$$f(x) \approx f(1) + f_x(1)(x - 1) + f_{xx}(1)(x - 1)^2 + (h.o.t.)$$
$$= 1^3 + 3 * 1^2(x - 1) + 6 * 1^3(x - 1)^2 + (h.o.t.)$$
$$= 1 + 3(x - 1) + 6(x - 1)^2 + (h.o.t.)$$

Taylor Expansion

Legend:
- f(x)
- $f(1) + f_x(1)(x-1)$
- $f(1) + f_x(1)(x-1) + f_{xx}(1)(x-1)^2$

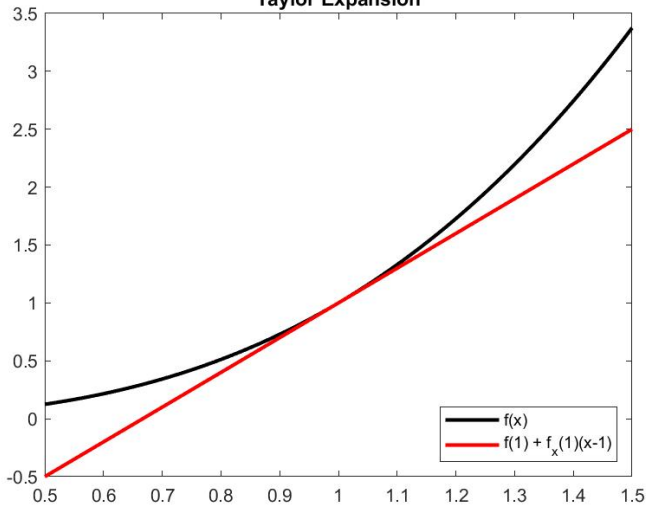Combined with log-approximation of percent changes...

$$f(x) \approx f(x_0) + f_x(x_0)x_0\hat{x} + f_{xx}(x_0)x_0^2\hat{x}^2 + (h.o.t.)$$

where $\hat{x} := \log(x) - \log(x_0)$

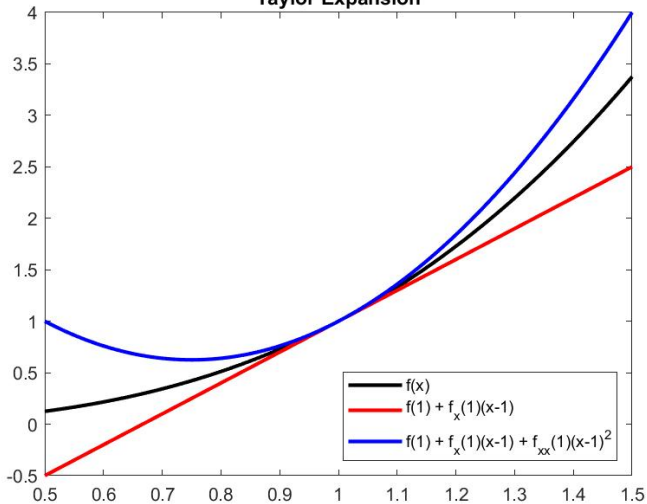Word of caution on Taylor expansion.

They are good "local" approximation, but may not
be good globally.

Taylor Expansion

Taylor Expansion

Legend:
- f(x)
- $f(1) + f_x(1)(x-1)$
- $f(1) + f_x(1)(x-1) + f_{xx}(1)(x-1)^2$

End of Math Background

Recall:

$$C_t^{-\chi_c} = \beta R_t C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1} \tag{1}$$

$$w_t = N_t^{\chi_n} C_t^{\chi_c} \tag{2}$$

$$\frac{Y_t}{C_t^{\chi_c}} \left[\varphi \left(\Pi_t - 1 \right) \Pi_t - (1-\theta)(1+\tau) - \theta w_t \right] \tag{3}$$

$$= \beta \frac{Y_{t+1}}{C_{t+1}^{\chi_c}} \varphi \left(\Pi_{t+1} - 1 \right) \Pi_{t+1} \tag{4}$$

$$Y_t = C_t + \frac{\varphi}{2} \left[\Pi_t - 1 \right]^2 Y_t \tag{5}$$

$$Y_t = N_t \tag{6}$$

$$R_t = \frac{1}{\beta} \Pi_t^{\phi_\pi} \tag{7}$$

$$U_t = \frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1+\chi_n}}{1+\chi_n} \tag{8}$$

Loglinearize the consumption Euler equation **around the zero-inflation steady state**:

$$C_t^{-\chi_c} R_t^{-1} = \beta \delta_t C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1}$$

$$LHS = f(C_t, R_t)$$
$$RHS = g(\delta_t, C_{t+1}, R_{t+1})$$

$$f_C = -\chi_c C_t^{-\chi_c-1} R_t^{-1}$$
$$f_R = -C_t^{-\chi_c} R_t^{-2}$$

$$g_\delta = \beta C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1}$$
$$g_C = -\chi_c \beta C_{t+1}^{-\chi_c-1} \Pi_{t+1}^{-1}$$
$$g_\Pi = -\beta C_{t+1}^{-\chi_c} \Pi_{t+1}^{-2}$$

$$LHS \approx f(C_{ss}, R_{ss}) + f_{C,ss}C_{ss}\hat{C}_t + f_{R,ss}R_{ss}\hat{R}_t$$
$$= f(C_{ss}, R_{ss}) + f(C_{ss}, R_{ss})\left[-\chi_c\hat{C}_t\right] + f(C_{ss}, R_{ss})\left[-\hat{R}_t\right]$$
$$RHS \approx g(\delta_{ss}, C_{ss}, \Pi_{ss}) + g_{\delta,ss}\delta_{ss}\hat{\delta}_t + g_{C,ss}C_{ss}\hat{C}_{t+1} + g_{\Pi,ss}\Pi_{ss}\hat{\Pi}_{t+1}$$
$$= g(\delta_{ss}, C_{ss}, \Pi_{ss}) + g(\delta_{ss}, C_{ss}, \Pi_{ss})\hat{\delta}_t$$
$$+ g(\delta_{ss}, C_{ss}, \Pi_{ss})\left[-\chi_c\hat{C}_{t+1}\right] + g(\delta_{ss}, C_{ss}, \Pi_{ss})\left[-\hat{\Pi}_{t+1}\right]$$

Using $f(C_{ss}, R_{ss}) = g(\delta_{ss}, C_{ss}, \Pi_{ss})$ and rearranging terms,

$$\hat{C}_t = \hat{C}_{t+1} - \frac{1}{\chi_c}\left[\hat{R}_t - \hat{\Pi}_{t+1} + \hat{\delta}_t\right]$$

Loglinearize the intratemporal optimality condition **around the zero-inflation steady state**:

$$w_t = N_t^{\chi_n} C_t^{\chi_c}$$

$$LHS = f(w_t)$$
$$RHS = g(C_t, N_t)$$

$$f_w = 1$$
$$g_C = \chi_c N_t^{\chi_n} C_t^{\chi_c - 1}$$
$$g_N = \chi_n N_t^{\chi_n - 1} C_t^{\chi_c}$$

$$
\begin{aligned}
LHS &\approx f(w_{ss}) + f_{w,ss} w_{ss} \hat{w}_t \\
&= f(w_{ss}) + f(w_{ss}) \hat{w}_t \\
RHS &\approx g(C_{ss}, N_{ss}) + g_{C,ss} C_{ss} \hat{C}_t + g_{N,ss} N_{ss} \hat{N}_t \\
&= g(C_{ss}, N_{ss}) + g(C_{ss}, N_{ss}) \left[\chi_c \hat{C}_t \right] + g(C_{ss}, N_{ss}) \left[\chi_N \hat{N}_t \right]
\end{aligned}
$$

Using $f(w_{ss}) = g(C_{ss}, N_{ss})$ and rearranging terms,

$$
\hat{w}_t = \chi_c \hat{C}_t + \chi_n \hat{N}_t
$$

Loglinearize the optimality condition of intermediate-goods producers **around the zero-inflation steady state**:

$$Y_t C_t^{-\chi_c} \left[\varphi \left(\Pi_t - 1 \right) \Pi_t - (1 - \theta)(1 + \tau) - \theta w_t \right]$$
$$= \beta Y_{t+1} C_{t+1}^{-\chi_c} \varphi \left(\Pi_{t+1} - 1 \right) \Pi_{t+1}$$

$$LHS = f(Y_t, C_t, \Pi_t, w_t)$$
$$RHS = g(Y_{t+1}, C_{t+1}, \Pi_{t+1})$$

$$f_Y = C_t^{-\chi_c} \left[\varphi \left(\Pi_t - 1 \right) \Pi_t - (1 - \theta)(1 + \tau) - \theta w_t \right]$$

$$f_C = -\chi_c Y_t C_t^{-\chi_c - 1} \left[\varphi \left(\Pi_t - 1 \right) \Pi_t - (1 - \theta)(1 + \tau) - \theta w_t \right]$$

$$f_\Pi = Y_t C_t^{-\chi_c} \varphi \left[2\Pi_t - 1 \right]$$

$$f_w = -Y_t C_t^{-\chi_c} \theta$$

$$g_Y = \beta C_{t+1}^{-\chi_c} \varphi \left(\Pi_{t+1} - 1 \right) \Pi_{t+1}$$

$$g_C = -\chi_c \beta Y_{t+1} C_{t+1}^{-\chi_c - 1} \varphi \left(\Pi_{t+1} - 1 \right) \Pi_{t+1}$$

$$g_\Pi = \beta Y_{t+1} C_{t+1}^{-\chi_c} \varphi \left[2\Pi_{t+1} - 1 \right]$$

Note that $\Pi_{ss} = 1$ and $w_{ss} = \frac{(\theta-1)(1+\tau)}{\theta}$ Thus,

$$f_{Y,ss} = 0$$
$$f_{C,ss} = 0$$
$$f_{\Pi,ss} = Y_{ss} C_{ss}^{-\chi_c} \varphi$$
$$f_{w,ss} = -Y_{ss} C_{ss}^{-\chi_c} \theta$$

$$g_{Y,ss} = 0$$
$$g_{C,ss} = 0$$
$$g_{\Pi,ss} = \beta Y_{ss} C_{ss}^{-\chi_c} \varphi$$

$$LHS \approx f(Y_{ss}, C_{ss}, \Pi_{ss}, w_{ss}) + f_{\Pi,ss}\Pi_{ss}\hat{\Pi}_t + f_{w,ss}w_{ss}\hat{w}_t$$
$$= f(Y_{ss}, C_{ss}, \Pi_{ss}, w_{ss}) + Y_{ss}C_{ss}^{-\chi_c}\varphi\Pi_{ss}\hat{\Pi}_t - Y_{ss}C_{ss}^{-\chi_c}\theta w_{ss}\hat{w}_t$$
$$RHS \approx g(Y_{ss}, C_{ss}, \Pi_{ss}) + g_{\Pi,ss}\Pi_{ss}\hat{\Pi}_{t+1}$$
$$= g(Y_{ss}, C_{ss}, \Pi_{ss}) + \beta Y_{ss}C_{ss}^{-\chi_c}\varphi\Pi_{ss}\hat{\Pi}_{t+1}$$

Using $f(Y_{ss}, C_{ss}, \Pi_{ss}, w_{ss}) = g(Y_{ss}, C_{ss}, \Pi_{ss})$, $\Pi_{ss} = 1$,
$w_{ss} = \frac{(\theta-1)(1+\tau)}{\theta}$ and rearranging terms,

$$Y_{ss}C_{ss}^{-\chi_c}\varphi\Pi_{ss}\hat{\Pi}_t - Y_{ss}C_{ss}^{-\chi_c}\theta w_{ss}\hat{w}_t = Y_{ss}C_{ss}^{-\chi_c}\varphi\Pi_{ss}\hat{\Pi}_{t+1}$$
$$\Leftrightarrow \varphi\hat{\Pi}_t - \theta\frac{(\theta-1)(1+\tau)}{\theta}\hat{w}_t = \beta\varphi\hat{\Pi}_{t+1}$$
$$\Leftrightarrow \hat{\Pi}_t = \frac{(\theta-1)(1+\tau)}{\varphi}\hat{w}_t + \beta\hat{\Pi}_{t+1}$$

Loglinearize the aggregate production function **around the zero-inflation steady state**.

$$Y_t = N_t$$

$$LHS = f(Y_t), RHS = g(N_t)$$

$$f_Y = 1, g_N = 1$$

$$LHS \approx f(Y_{ss}) + f_{Y,ss} Y_{ss} \hat{Y}_t$$
$$= f(Y_{ss}) + f(Y_{ss}) \hat{Y}_t$$
$$RHS \approx g(N_{ss}) + g_{N,ss} N_{ss} \hat{N}_t$$
$$= g(N_{ss}) + g(N_{ss}) \hat{N}_t$$

Using $f(Y_{ss}) = g(N_{ss})$, we get

$$\hat{Y}_t = \hat{N}_t$$

Loglinearize the aggregate resource constraint **around the zero-inflation steady state**:

$$Y_t = C_t + \frac{\varphi}{2} \left[\Pi_t - 1\right]^2 Y_t$$

$$
\begin{aligned}
LHS &= f(Y_t) \\
RHS &= g(C_t, Y_t, \Pi_t) \\
f_Y &= 1 \\
g_C &= 1 \\
g_Y &= \frac{\varphi}{2} \left[\Pi_t - 1\right]^2 \\
g_\Pi &= \varphi \left[\Pi_t - 1\right] Y_t
\end{aligned}
$$

Note that $g_{Y,ss} = g_{\Pi,ss} = 0$ because $\Pi_{ss} = 1$.

$$LHS \approx f(Y_{ss}) + f_{Y,ss} Y_{ss} \hat{Y}_t$$
$$= f(Y_{ss}) + Y_{ss} \hat{Y}_t$$
$$RHS \approx g(C_{ss}, Y_{ss}, \Pi_{ss}) + g_{C,ss} C_{ss} \hat{C}_t$$
$$= g(C_{ss}, Y_{ss}, \Pi_{ss}) + C_{ss} \hat{C}_t$$

Note $f(Y_{ss}) = g(C_{ss}, Y_{ss}, \Pi_{ss})$. Also, $Y_{ss} = C_{ss}$ because $\Pi_{ss} = 1$. We obtain

$$\hat{Y}_t = \hat{C}_t$$

Loglinearizing the Taylor rule **around the zero-inflation steady state**:

$$R_t = \max \left[1, \frac{1}{\beta} \Pi_t^\phi \right]$$

$$LHS = f(R_t)$$
$$RHS = g(\Pi_t)$$

$$f_R = 1$$
$$g_\Pi = \phi \frac{1}{\beta} \Pi_t^{\phi-1}$$

$$LHS \approx f(R_{ss}) + f_{R,ss}R_{ss}\hat{R}_t$$
$$= f(R_{ss}) + f(R_{ss})\hat{R}_t$$
$$RHS \approx g(\Pi_{ss}) + g_{\Pi,ss}\Pi_{ss}\hat{\Pi}_t$$
$$= g(C_{ss}, Y_{ss}, \Pi_{ss}) + g(C_{ss}, Y_{ss}, \Pi_{ss})\phi\hat{\Pi}_t$$

$$\hat{R}_t = \max\left[1 - \frac{1}{\beta}, \phi\hat{\Pi}_t\right]$$

So far, we have

$$\hat{C}_t = \hat{C}_{t+1} - \frac{1}{\chi_c}\left[\hat{R}_t - \hat{\Pi}_{t+1} + \hat{\delta}_t\right]$$

$$\hat{w}_t = \chi_c\hat{C}_t + \chi_n\hat{N}_t$$

$$\hat{\Pi}_t = \frac{(\theta-1)(1+\tau)}{\varphi}\hat{w}_t + \beta\hat{\Pi}_{t+1}$$

$$\hat{Y}_t = \hat{N}_t$$

$$\hat{Y}_t = \hat{C}_t$$

$$\hat{R}_t = \max\left[1 - \frac{1}{\beta}, \phi\hat{\Pi}_t\right]$$

Eliminating $\hat{w}_t$, $\hat{N}_t$, and $\hat{C}_t$, we obtain

$$\hat{Y}_t = \hat{Y}_{t+1} - \frac{1}{\chi_c}\left[\hat{R}_t - \hat{\Pi}_{t+1} + \hat{\delta}_t\right] \quad [\text{"IS Curve"}]$$

$$\hat{\Pi}_t = \frac{(\theta-1)(1+\tau)}{\varphi}(\chi_c + \chi_n)\hat{Y}_t + \beta\hat{\Pi}_{t+1} \quad [\text{"Phillips Curve"}]$$

$$\hat{R}_t = \max\left[1 - \frac{1}{\beta}, \phi\hat{\Pi}_t\right]$$

Change of notation: Using

$$y_t := \hat{Y}_t, \quad \pi_t := \hat{\Pi}_t, \quad i_t := \hat{R}_t + r^*$$
$$r^* := \frac{1}{\beta} - 1, \quad \sigma := \frac{1}{\chi_c}, \quad \kappa := \frac{(\theta - 1)(1 + \tau)}{\varphi}(\chi_c + \chi_n)$$

we obtain

$$y_t = y_{t+1} - \sigma \left[i_t - \pi_{t+1} - r^* \right]$$
$$\pi_t = \kappa y_t + \beta \pi_{t+1}$$
$$i_t = \max \left[0, r^* + \phi \pi_t \right]$$

Taylor-rule equilibrium:

$$y_t = y_{t+1} - \sigma \left[i_t - \pi_{t+1} - r^* \right]$$
$$\pi_t = \kappa y_t + \beta \pi_{t+1}$$
$$i_t = r^* + \phi \pi_t$$

Implications of the IS curve and the Phillips curve

▶ 1. Output today depends on the sum of future real rates.

$$
\begin{aligned}
y_t &= y_{t+1} - \sigma \left[i_t - \pi_{t+1} - r^* \right] \\
&= \left[y_{t+2} - \sigma \left[i_{t+1} - \pi_{t+2} - r^* \right] \right] - \sigma \left[i_t - \pi_{t+1} - r^* \right] \\
&= \left[y_{t+3} - \sigma \left[i_{t+2} - \pi_{t+3} - r^* \right] \right] \\
&\quad - \sigma \left[i_{t+1} - \pi_{t+2} - r^* \right] - \sigma \left[i_t - \pi_{t+1} - r^* \right] \\
&\quad ... \\
&= -\sigma \sum_{k=0}^{\infty} \left[i_{t+k} - \pi_{t+k+1} - r_{t+k}^* \right]
\end{aligned}
$$

using $y_\infty = y_{ss} = 0$.

- 2. Inflation today depends on the sum of output (which is proportional to real wages—marginal costs).

$$
\begin{aligned}
\pi_t &= \kappa y_t + \beta \pi_{t+1} \\
&= \kappa y_t + \beta \left[\kappa y_{t+1} + \beta \pi_{t+2} \right] \\
&= \kappa y_t + \beta \kappa y_{t+1} + \beta^2 \left[\kappa y_{t+2} + \beta \pi_{t+3} \right] \\
&= \kappa y_t + \beta \kappa y_{t+1} + \beta^2 \kappa y_{t+2} + ... \\
&= ... \\
&= \kappa \sum_{k=0}^{\infty} \beta^k y_{t+k}
\end{aligned}
$$

Plan

- First-Order Approximation of the Equilibrium Conditions

- **Second-Order Approximation of the Household's Welfare**

Take the second-order expansion of the household utility **around the zero-inflation steady state**.

$$\begin{aligned}
U(C_t, N_t) &\approx U_{ss} + U_{C,ss} C_{ss} \hat{C}_t + U_{N,ss} N_{ss} \hat{N}_t \\
&+ \frac{1}{2} U_{CC,ss} C_{ss}^2 \hat{C}_t^2 + \frac{1}{2} U_{CN,ss} C_{ss} N_{ss} \hat{C}_t \hat{N}_t \\
&+ \frac{1}{2} U_{NC,ss} N_{ss} C_{ss} \hat{N}_t \hat{C}_t + \frac{1}{2} U_{NN,ss} N_{ss}^2 \hat{N}_t^2
\end{aligned}$$

Useful stuff (I):

$$U_{C,ss} = C_t^{-\chi_c}, \quad U_{N,ss} = -N_t^{\chi_n}, \quad U_{CC,ss} = -\chi_c C_t^{-\chi_c-1},$$

$$U_{CN,ss} = 0, \quad U_{NC,ss} = 0, \quad U_{NN,ss} = -\chi_n N_t^{\chi_n-1}$$

$$\frac{U_{CC,ss}}{U_{C,ss}} = \frac{-\chi_c C_{ss}^{-\chi_c-1}}{C_{ss}^{-\chi_c}} = -\chi_c C_{ss}^{-1}$$

$$\frac{U_{NN,ss}}{U_{N,ss}} = \frac{-\chi_n N_t^{\chi_n-1}}{-N_t^{\chi_n}} = \chi_n N_{ss}^{-1}$$

Useful stuff (II): Second-order approximation of the aggregate resource constraint.

$$N_t = C_t + \frac{\varphi}{2}(\Pi_t - 1)^2 N_t$$

$$LHS = f(N_t)$$
$$RHS = g(C_t, N_t, \Pi_t)$$

$$f_N = 1, \quad f_{NN} = 0$$

$$g_C = 1, \quad g_N = \frac{\varphi}{2}(\Pi_t - 1)^2, \quad g_\Pi = \varphi(\Pi_t - 1)N_t$$

$$g_{CC} = 0, \quad g_{CN} = 0, \quad g_{C\Pi} = 0$$

$$g_{NC} = 0, \quad g_{NN} = 0, \quad g_{N\Pi} = \varphi(\Pi_t - 1)$$

$$g_{\Pi C} = 0, \quad g_{\Pi N} = \varphi(\Pi_t - 1), \quad g_{\Pi\Pi} = \varphi N_t$$

For g, at the steady state, the only non-zero coefficient is $g_C = 1$ and $g_{\Pi\Pi} = \varphi N_{ss}$ (because $\Pi_{ss} = 1$).

$$
\begin{aligned}
LHS &\approx f_{ss} + f_{N,ss} N_{ss} \hat{N}_t + \frac{f_{N,ss}}{2} N_{ss}^2 \hat{N}_t^2 \\
&= f_{ss} + N_{ss} \hat{N}_t \\
RHS &\approx g_{ss} + g_{C,ss} C_{ss} \hat{C}_t + g_{N,ss} N_{ss} \hat{N}_t + g_{\Pi,ss} \Pi_{ss} \hat{\Pi}_t \\
&\quad + \frac{1}{2}\Big[g_{CC,ss} C_{ss}^2 \hat{C}_t^2 + g_{CN,ss} C_{ss} N_{ss} \hat{C}_t \hat{N}_t + g_{C\Pi,ss} \Pi_{ss}^2 \hat{\Pi}_t^2 \\
&\quad + g_{NC,ss} N_{ss} C_{ss} \hat{N}_t \hat{C}_t + g_{NN,ss} N_{ss}^2 \hat{N}_t^2 + g_{N\Pi,ss} N_{ss} \Pi_{ss} \hat{N}_t \hat{\Pi}_t \\
&\quad + g_{\Pi C,ss} \Pi_{ss} C_{ss} \hat{\Pi}_t \hat{C}_t + g_{\Pi N,ss} \Pi_{ss} N_{ss} \hat{\Pi}_t \hat{N}_t + g_{\Pi\Pi,ss} \Pi_{ss}^2 \hat{\Pi}_t^2 \Big] \\
&= g_{ss} + g_{C,ss} C_{ss} \hat{C}_t + \frac{1}{2} g_{\Pi\Pi,ss} \Pi_{ss}^2 \hat{\Pi}_t^2 \\
&= g_{ss} + C_{ss} \hat{C}_t + \frac{\varphi}{2} N_{ss} \hat{\Pi}_t^2
\end{aligned}
$$

$$
\Rightarrow \hat{N}_t = \hat{C}_t + \frac{\varphi}{2} \hat{\Pi}_t^2
$$

$$
\begin{aligned}
U_t - U_{ss} \approx{} & U_{C,ss} C_{ss} \hat{C}_t + U_{N,ss} N_{ss} \hat{N}_t \\
& + \frac{1}{2} U_{CC,ss} C_{ss}^2 \hat{C}_t^2 + \frac{1}{2} U_{NN,ss} N_{ss}^2 \hat{N}_t^2 \\
={} & U_{C,ss} C_{ss} \left(\hat{C}_t + \frac{1}{2} \frac{U_{CC,ss} C_{ss}^2}{U_{C,ss} C_{ss}} \hat{C}_t^2 \right) \\
& + U_{N,ss} N_{ss} \left(\hat{N}_t + \frac{1}{2} \frac{U_{NN,ss} N_{ss}^2}{U_{N,ss} N_{ss}} \hat{N}_t^2 \right) \\
={} & U_{C,ss} C_{ss} \left(\hat{C}_t - \frac{1}{2} \chi_c \hat{C}_t^2 \right) \\
& + U_{N,ss} N_{ss} \left(\hat{N}_t + \frac{1}{2} \chi_n \hat{N}_t^2 \right)
\end{aligned}
$$

$$\frac{U_t - U_{ss}}{U_{C,ss} C_{ss}} \approx \left(\hat{C}_t - \frac{\chi_c}{2} \hat{C}_t^2 \right) + \frac{U_{N,ss} N_{ss}}{U_{C,ss} C_{ss}} \left(\hat{N}_t + \frac{\chi_n}{2} \hat{N}_t^2 \right)$$

$$= \left(\hat{C}_t - \frac{\chi_c}{2} \hat{C}_t^2 \right) - w_{ss} \left(\hat{N}_t + \frac{\chi_n}{2} \hat{N}_t^2 \right)$$

$$= \left(\hat{C}_t - \frac{\chi_c}{2} \hat{C}_t^2 \right) - \frac{(\theta - 1)(1 + \tau)}{\theta} \left(\hat{N}_t + \frac{\chi_n}{2} \hat{N}_t^2 \right)$$

$$= \left(\hat{C}_t - \frac{\chi_c}{2} \hat{C}_t^2 \right)$$

$$- \frac{(\theta - 1)(1 + \tau)}{\theta} \left(\hat{C}_t + \frac{\varphi}{2} \hat{\Pi}^2 + \frac{\chi_n}{2} \left[\hat{C}_t + \frac{\varphi}{2} \hat{\Pi}^2 \right] \right)$$

$$\approx \left(\hat{C}_t - \frac{\chi_c}{2} \hat{C}_t^2 \right) - \frac{(\theta-1)(1+\tau)}{\theta} \left(\hat{C}_t + \frac{\varphi}{2} \hat{\Pi}^2 + \frac{\chi_n}{2} \hat{C}_t^2 \right)$$

$$= \frac{\theta - (\theta-1)(1+\tau)}{\theta} \hat{C}_t$$

$$- \left(\frac{\chi_c}{2} + \frac{(\theta-1)(1+\tau)}{\theta} \frac{\chi_n}{2} \right) \hat{C}_t^2 - \frac{(\theta-1)(1+\tau)}{\theta} \frac{\varphi}{2} \hat{\Pi}^2$$

$$\propto \frac{\theta - (\theta-1)(1+\tau)}{\varphi(\theta-1)(1+\tau)} \hat{C}_t - \frac{1}{2} \left[\hat{\Pi}^2 + \frac{\theta \chi_c + (\theta-1)(1+\tau)\chi_n}{\varphi(\theta-1)(1+\tau)} \hat{C}_t^2 \right]$$

$$= -\frac{1}{2} \left[\hat{\Pi}^2 + \frac{\chi_c + \chi_n}{\varphi} \hat{C}_t^2 \right] \quad \text{(if } \tau = \frac{1}{\theta-1}\text{)}$$

Change of notation: Using

$$y_t := \hat{Y}_t, \quad \pi_t := \hat{\Pi}_t, \quad i_t := \hat{R}_t + r^*$$

$$r^* := \frac{1}{\beta} - 1, \quad \sigma := \frac{1}{\chi_c}, \quad \kappa := \frac{\theta - 1}{\varphi}(\chi_c + \chi_n)$$

we obtain

$$\frac{U_t - U_{ss}}{U_{C,ss}C_{ss}}$$

$$\propto \frac{\theta - (\theta - 1)(1 + \tau)}{\varphi(\theta - 1)(1 + \tau)} y_t - \frac{1}{2}\left[\pi^2 + \frac{\theta\chi_c + (\theta - 1)(1 + \tau)\chi_n}{\varphi(\theta - 1)(1 + \tau)} y_t^2\right]$$

$$= -\frac{1}{2}\left[\pi^2 + \frac{\chi_c + \chi_n}{\varphi} y_t^2\right] \quad (\text{if } \tau = \frac{1}{\theta - 1})$$